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Numerical Solution of Linear Stochastic Differential Equations

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Abstract—A new algorithm of first order is proposed for the numerical solution of linear Ito stochastic differential equations. The error of the scheme is studied for the scalar case analytically and by the Monte Carlo method. It turns out that the new scheme gives better results.

A new simple form of the Runge-Kutta method is derived.

Keywords—Stochastic differential equation, Numerical solution, Monte Carlo method, Runge-Kutta method.

1. INTRODUCTION

We consider a linear Ito stochastic differential equation (SDE) with constant coefficients

$$dX_t = AX_t\,dt + B\,dW_t, \tag{1}$$

for $0 \le t \le T$ with a random initial value X_0. Here the diffusion process $\{X_t; t \in [0,T]\}$ is a p-dimensional vector, the drift coefficient A is a $p \times p$ matrix, the diffusion coefficient B is a $p \times q$ matrix and $\{W_t; t \in [0,T]\}$ is a q-dimensional standard Wiener process with independent components. We suppose that throughout the paper X_0 is independent of $\{\mathcal{F}_t^W\} = \sigma\{W_s, s \le t\}$.

We shall use a partition of $[0,T]$ with division points $0 = t_0, t_1, \dots, t_k = T$, where $t_k = kh$, $k = 0,1,\dots,K$, $h = \frac{T}{K}$ and K is an arbitrary positive integer. Further, let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_K$ be i.i.d.r. vectors, $\mathbf{e}_k \approx \mathcal{N}(0, \mathbf{I})$ and $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \dots, \boldsymbol{\varepsilon}_K$ be i.i.d.r. vectors, $\boldsymbol{\varepsilon}_k \approx \mathcal{N}(0, B_\varepsilon)$, where

$$B_\varepsilon = \int_0^h e^{A(h-s)}\,BB'e^{A'(h-s)}\,ds. \tag{2}$$

The exact solution of SDE (1) is (see, e.g., [1])

$$X_t = e^{At}\,X_0 + e^{At}\int_0^t e^{-As}B\,dW_s. \tag{3}$$

In the case of simulation X_t the stochastic Ito integral in (3) must be evaluated by numerical techniques. Using another way of simulation X_t, we get from [2]: let $X_k = X_{t_k}$, $k = 1,2,\dots,K$, then

$$X_k = e^{Ah}X_{k-1} + \boldsymbol{\varepsilon}_k = e^{Akh}X_0 + \sum_{i=1}^{k} e^{A(k-i)h}\boldsymbol{\varepsilon}_i. \tag{4}$$

The author would like to thank P. Mandl for pointing out the problem of estimating constants in error terms of approximating processes for SDE. Special thanks are also due to M. Arato, who called my attention to relation (4).

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In this case, before the generation $\varepsilon_1, \dots, \varepsilon_K$, we must evaluate B_ε from (2). The third way of simulation X_t is via numerical solution of SDE (1) by iterative schemes. This is the way we will follow.

Let $\{\hat{X}_k\}_{k=0,K}$ be an arbitrary approximating process. We will investigate the error of the process $\{\hat{X}_k\}$ at the terminal instant $t = T$. We will say (see [3]) that an approximating process $\{\hat{X}_k\}_{k=0,K}$ converges in the strong sense with order $\delta > 0$ (the error of a numerical solution $\hat{X}_K$ for X_T is of order δ) if there exists a constant $0 < C < \infty$ such that

$$E\|X_T - \hat{X}_K\| \le Ch^\delta, \tag{5}$$

for any $h \in (0,1]$, where $\|X\|^2 = \text{tr } XX'$.

We introduce

$$W_k = W_{t_k}, \quad \Delta W_k = W_k - W_{k-1}, \quad k = 1, 2, \dots, K.$$

It is known from Milstejn and Pjanzin [4] that the Euler scheme

$$\begin{aligned} X_0^e &= X_0, \\ X_{k+1}^e &= X_k^e + AX_k^e h + B\,\Delta W_{k+1}, \qquad k = 0, 1, \dots, K-1, \end{aligned} \tag{6}$$

converges to the solution of SDE (1) with order 1.

From (6) the numerical solution for X_T can be easily derived as follows

$$X_K^e = (Ah + I)^K X_0 + \sum_{k=1}^{K} (Ah + I)^{K-k} B\,\Delta W_k. \tag{7}$$

The integration of (1) by trapezoidal rule yields the iterative scheme

$$\begin{aligned} X_0^b &= X_0, \\ X_{k+1}^b &= X_k^b + \frac{1}{2}\left(A\left(X_k^b + X_{k+1}^b\right)h\right) + B\Delta W_{k+1}, \qquad k = 0, 1, \dots, K-1. \end{aligned}$$

Hence, we get for X_T a new numerical solution

$$\begin{aligned} X_K^b = {} & \left(\left(\frac{I - Ah}{2}\right)^{-1}\left(\frac{I + Ah}{2}\right)\right)^K X_0 \\ & + \left(\frac{I - Ah}{2}\right)^{-1} \sum_{k=1}^{K}\left(\left(\frac{I - Ah}{2}\right)^{-1}\left(\frac{I + Ah}{2}\right)\right)^{K-k} B\,\Delta W_k. \end{aligned} \tag{8}$$

In Section 2, we prove that both X_K^e and X_K^b are of order 1. Section 3 is devoted to the estimation of constant C in (5) for schemes (7) and (8) in scalar case of (1). By the comparison of these estimates the numerical solution (7) seems to be preferable. In Section 4 the two solutions (7) and (8) are investigated by the Monte Carlo method. The simulation results also show that (8) gives in mean better solution. In Section 5 we give a simple form of the Runge-Kutta scheme of order M for the SDE (1).

2. VECTOR SDE

In this section, we show that, for (7) and (8), the error over a finite interval $[0, T]$ is of order 1.

THEOREM 1. *Let $E\|X_0\|^2 = \text{tr } B_0 < \infty$. Then*

(i)
$$E\|X_T - X_K^e\|^2 = O(h^2),$$

(ii)
$$E\|X_T - X_K^b\|^2 = O(h^2),$$

where X_T, X_K^e and X_K^b are defined by (3), (7) and (8), respectively.

PROOF.

(i) Since

$$\int_0^h e^{-As}BB'e^{-A's}\,ds = h\,e^{-Ah}BB'e^{-A'h} + O(h^2),$$

and $B\Delta W_k = \sqrt{h}\,B\,\mathbf{e}_k$, from (4), (7) we get

$$X_T = X_K = e^{AT}\left(X_0 + \sum_{k=1}^{K} e^{-Akh}\left(\sqrt{h}B + IO\left(h^{3/2}\right)\right)\mathbf{e}_k\right),$$

$$X_K^e = (I + Ah)^K \left(X_0 + \sum_{k=1}^{K} (I + Ah)^{-k}\sqrt{h}\,B\mathbf{e}_k\right).$$

We have further

$$\left(e^{Ah}\right)^K - (I + Ah)^K = \left(I + Ah + IO(h^2)\right)^K - (I + Ah)^K = IKO(h^2) = IO(h),$$

and similarly $\max\limits_{0\le k\le K}\left\{e^{Akh} - (I + Ah)^k\right\} = IO(h)$. Hence, by the independentness of $\mathbf{e}_k$, $k = 1, 2\dots, K$ and X_0, we get

$$\begin{aligned}
E\left\|X_T - X_K^e\right\|^2 &\le \left\|e^{AT} - (I + Ah)K\right\|^2 E\left\|X_0\right\|^2 \\
&\quad + \sum_{k=1}^{K}\left\|e^{A(K-k)h}\left(\sqrt{h}B + IO\left(h^{3/2}\right)\right) - (I + Ah)^{K-k}\sqrt{h}B\right\|^2 \\
&\le O(h^2) + Kh\max_{0\le k\le K-1}\left\|e^{Akh}(B + IO(h)) - (I + Ah)^k B\right\|^2 = O(h^2).
\end{aligned}$$

(ii) Since $(I - Ah/2)^{-1} = I + Ah/2 + IO(h^2)$, the proof is similar to (i). ∎

3. SCALAR SDE

In the previous section, we proved that the approximations (7) and (8) are of first order, i.e., $E\left\|X_T - X_K^e\right\| \le C_e h$ and $E\left\|X_T - X_K^b\right\| \le C_b h$. In this section, we give upper bounds for C_e and C_b in the scalar case of SDE (1)

$$dx_t = ax_t\,dt + b\,dw_t. \tag{9}$$

The exact solution of (9) is (see (3), (4))

$$\begin{aligned}
x_T &= e^{aT}\left(x_0 + \int_0^T e^{-as}\,dw_s\right) \\
&= e^{aT}\left(x_0 + \sum_{k=1}^{K} e^{-akh}\,\varepsilon_k\right).
\end{aligned} \tag{10}$$

For the Euler scheme,

$$x_{k+1}^e = x_k^e + ahx_k^e + b\Delta w_{k+1}, \qquad k = 0, 1, \dots, K-1,$$

we get a numerical solution for x_T

$$x_K^e = (ah + 1)^K x_0 + b(ah + 1)^K \sum_{k=1}^{K} (ah + 1)^{-k}\,\Delta w_k. \tag{11}$$

On integration (9) by trapezoidal rule, we receive

$$x_{k+1}^b = x_k^b + \frac{1}{2}\left(a(x_k^b + x_{k+1}^b)h\right) + b\,\Delta w_{k+1}, \qquad k = 0, 1, \dots, K-1,$$

whence we get, for x_T, a numerical solution

$$x_K^b = \left(\frac{1+ah/2}{1-ah/2}\right)^K x_0 + \left(\frac{1+ah/2}{1-ah/2}\right)^K \frac{b}{1-ah/2} \sum_{k=1}^K \left(\frac{1+ah/2}{1-ah/2}\right)^{-k} \Delta w_k.$$

It is well known that the second moment of x_K is

$$Ex_K^2 = e^{2aT}\left(Ex_0^2 + \frac{b^2}{2a}\right) - \frac{b^2}{2a}.$$

The second moments of x_K^e and x_K^b, we get from the following lemma.

LEMMA 1. *Let $Ex_0^2 = b_0 < \infty$. Then*

(i)

$$E\left(x_K^e\right)^2 = \left(1+ah\right)^{2K} b_0 + \frac{b^2}{2a} \frac{\left(1+ah\right)^{2K} - 1}{1+ah/2};$$

(ii)

$$E\,x_K\,x_K^e = e^{aT}\left(1+ah\right)^K b_0 + \frac{b^2}{2a} \frac{\sqrt{2ah\left(e^{2ah}-1\right)}}{\left(1+ah\right)e^{ah} - 1}\left(\left(1+ah\right)^K e^{aT} - 1\right);$$

(iii)

$$E\left(x_K^b\right)^2 = \left(\frac{1+(ah/2)}{1-(ah/2)}\right)^{2K}\left(b_0 + \frac{b^2}{2a}\right) - \frac{b^2}{2a};$$

(iv)

$$\begin{aligned} E\,x_K\,x_K^b = {} & e^{aT}\left(\frac{1+(ah/2)}{1-(ah/2)}\right)^K b_0 \\ & + \frac{b^2}{2a} \frac{\sqrt{2ah\left(e^{2ah}-1\right)}}{e^{ah}(1+(ah/2)) - 1 + (ah/2)}\left(e^{aT}\left(\frac{1+(ah/2)}{1-(ah/2)}\right)^K - 1\right). \end{aligned}$$

PROOF.

(i) On account of the independence of $\Delta w_k, k = 1, 2, \dots, K$ and x_0, we obtain from (11)

$$E\left(x_K^e\right)^2 = \left(1+ah\right)^{2K} b_0 + b^2 h\left(1+ah\right)^{2K} \sum_{k=1}^K \left(1+ah\right)^{-2k}.$$

Since $\sum_{k=1}^K z^k = \frac{1-z^K}{1-z} z$, we get

$$\left(1+ah\right)^{2K} \sum_{k=1}^K \left(\left(1+ah\right)^{-2}\right)^k = \frac{\left(1+ah\right)^{2K} - 1}{\left(1+ah\right)^2 - 1},$$

whence the proof of (i) follows immediately.

Items (ii)–(iv) can be proved in the same way as (i). We mention only that, from (2) after integration, we get $E\varepsilon_k^2 = b_\varepsilon = \frac{b^2}{2a}(e^{2ah}-1)$, $k = 1,2,\dots,K$, and hence, e.g.,

$$Eb\,\Delta w_k \varepsilon_k = \sqrt{b^2 h b_\varepsilon} = \frac{b^2}{2a}\sqrt{2ah\left(e^{2ah}-1\right)}. \qquad \blacksquare$$

LEMMA 2.

(i) *If* $(|ah|/2) < 1$ *then*

$$\sqrt{2ah\left(e^{2ah}-1\right)} \ge \begin{cases} 2ah\,(1+(ah/2)), & a > 0, \\ -2ah\,(1+(ah/2)), & a < 0; \end{cases}$$

(ii) *if* $|ah| < 1$ *then*

$$e^{aT} - (1+ah)^K = e^{aT}\,\frac{a^2}{2}\,Th + O\left(h^2\right);$$

(iii) *if* $(|ah|/2) < 1$ *then*

$$e^{aT} - \left(\frac{1+(ah/2)}{1-(ah/2)}\right)^K = O\left(h^2\right).$$

PROOF.

(i) By the Taylor expansion, we get

$$\begin{aligned}\sqrt{2ah\left(e^{2ah}-1\right)} &= \left(2ah\left(2ah + 2a^2h^2 + \frac{8}{6}a^3h^3\right) + O\left(h^5\right)\right)^{0.5} \\ &= \left(\left(2ah + a^2h^2\right)^2 + \frac{5}{3}a^4h^4 + O\left(h^5\right)\right)^{0.5} \ge \left|2ah + a^2h^2\right|;\end{aligned}$$

(ii) Since $\ln(1+ah) = ah - \left((ah)^2/2\right) + O(h^3)$, we get

$$e^{aT} - (1+ah)^K = e^{aT}\left(1 - e^{-(a^2/2)Th + O(h^2)}\right) = e^{aT}\,\frac{a^2}{2}\,Th + O\left(h^2\right);$$

(iii) is proved by a complete analogy with (ii). $\blacksquare$

THEOREM 2. *Let* $b_0 < \infty$ *and* $|ah| < 1$. *Then*

$$E\left(x_T - x_K^e\right)^2 \le \left(\left(b_0 + \frac{b^2}{2a}\right)e^{2aT}\frac{a^4}{4}T^2 + e^{2aT}\frac{(ab)^2}{8}T + \frac{b^2}{2a}\left(e^{2aT}-1\right)\frac{13}{24}a^2\right)h^2 + O\left(h^3\right).$$

PROOF. Since

$$\begin{aligned}\frac{1}{1+(ah/2)} &= 1 - \frac{ah}{2} + \frac{a^2h^2}{4} + O(h^3), \qquad \text{and} \\ \frac{1}{(1+ah)e^{ah}-1} &= \frac{1}{2ah}\left(1 - \frac{3}{4}ah + \frac{11}{48}a^2h^2 + O(h^3)\right),\end{aligned}$$

by Lemma 1 (i), (ii) and Lemma 2 (i), (ii), we get

$$\begin{aligned}E\left(x_T - x_K^e\right)^2 &= Ex_K^2 + E\left(x_K^e\right)^2 - 2Ex_Kx_K^e \\ &= b_0\left(e^{aT} - (1+ah)^K\right)^2 \\ &\quad + \frac{b^2}{2a}\left(e^{2aT} - 1 + \frac{(1+ah)^{2K}-1}{1+(ah/2)} - 2\frac{\sqrt{2ah\left(e^{2ah}-1\right)}}{(1+ah)e^{ah}-1}\left((1+ah)^Ke^{aT}-1\right)\right) \\ &\le b_0\,e^{2aT}\,\frac{a^4}{4}\,T^2h^2 + \frac{b^2}{2a}\left(e^{2aT} - 1 + (1+ah)^{2K} - 1\right) \\ &\quad + \frac{b^2}{2a}\Bigg[\left((1+ah)^{2K}-1\right)\left(-\frac{ah}{2} + \frac{a^2h^2}{4} + O(h^3)\right) \\ &\quad - 2\left((1+ah)^Ke^{aT}-1\right) + \left(\frac{ah}{2} + \frac{7}{24}a^2h^2 + O(h^3)\right)\left((1+ah)^Ke^{aT}-1\right)\Bigg].\end{aligned}$$

We have further

$$(1+ah)^{2K}-1=e^{2aT}-1+O(h)=(1+ah)^{K}e^{aT}-1.$$

Hence, by Lemma 2 (ii), we obtain

$$\begin{aligned} E\big(x_T-x_K^e\big)^2 \le{} & \left(b_0+\frac{b^2}{2a}\right)e^{2aT}\frac{a^4}{4}T^2h^2 \\ & +\frac{b^2}{2a}\left(-\frac{1}{2}ah\Big((1+ah)^{2K}-1-(1+ah)^{K}e^{aT}+1\Big)\right) \\ & +\frac{b^2}{2a}\left(e^{2aT}-1\right)\frac{13}{24}a^2h^2+O(h^3), \end{aligned}$$

whence the proof follows immediately. ∎

Corollary 1. *In the case of stationary $\{x_t\}$, when $a<0$, $x_0\approx\mathcal{N}(0,b_0)$ and $b_0=-b^2/2a$, we have*

$$E\big(x_T-x_K^e\big)^2\le\left(e^{2aT}\frac{(ab)^2}{8}T+\frac{b^2}{2a}\big(e^{2aT}-1\big)\frac{13}{24}a^2\right)h^2+O(h^3).$$

Theorem 3. *Let $b_0<\infty$ and $(|ah|/2)<1$. Then*

$$E\big(x_T-x_K^b\big)^2\le\frac{ab^2}{12}\left(e^{2aT}-1\right)h^2+O(h^3).$$

Proof. From Lemma 2 (i), we get

$$-2\frac{\sqrt{2ah\left(e^{2ah}-1\right)}}{e^{ah}(1+(ah/2))-1+(ah/2)}\le-2\frac{1+(ah/2)}{1+(ah/2)+(5(ah)^2/24)+O(h^3)}=-2+\frac{a^2h^2}{6}+O(h^3),$$

and therefore, by Lemma 1 (iii), (iv) and Lemma 2 (iii), we obtain

$$\begin{aligned} E\big(x_T-x_K^b\big)^2 &= Ex_K^2+E\big(x_K^b\big)^2-2E\,x_k\,x_k^b \\ &\le\left(e^{2aT}+\left(\frac{1+(ah/2)}{1-(ah/2)}\right)^{2K}\right)\left(b_0+\frac{b^2}{2a}\right)-2\frac{b^2}{2a}-2e^{aT}\left(\frac{1+(ah/2)}{1-(ah/2)}\right)^{K}b_0 \\ &\quad+\frac{b^2}{2a}\left(-2+\frac{a^2h^2}{6}+O(h^3)\right)\left(e^{aT}\left(\frac{1+(ah/2)}{1-(ah/2)}\right)^{K}-1\right) \\ &=\left(e^{aT}-\left(\frac{1+(ah/2)}{1-(ah/2)}\right)^{K}\right)^2\left(b_0+\frac{b^2}{2a}\right) \\ &\quad+\frac{ab^2}{12}\left(e^{aT}\left(\frac{1+(ah/2)}{1-(ah/2)}\right)^{K}-1\right)h^2+O(h^3) \\ &=\frac{ab^2}{12}\left(e^{2aT}-1\right)h^2+O(h^3). \end{aligned}$$

∎

From this theorem, we can see that the error of a numerical solution x_K^b has the same form both for the stationary and the nonstationary case.

Example. Consider the scalar SDE Ito (9) with coefficients $a=-1$, $(a=1)$, $b=1$, $T=1.2$ and $x_0\approx\mathcal{N}(0,0.5)$ Then from Corollary 1 and Theorem 3, we get

$$E\big(x_T-x_K^e\big)^2\le0.510^2h^2+O(h^3),\qquad\left(E\big(x_T-x_K^e\big)^2\le2.877^2h^2+O(h^3)\right),$$

and

$$E\big(x_T-x_K^b\big)^2\le0.275h^2+O(h^3),\qquad\left(E\big(x_T-x_K^b\big)^2\le0.914^2h^2+O(h^3)\right).$$

The computed constants $C_e=0.51$ (2.887), $C_b=0.275$ (0.914) may be used in the choice of discretization step h. Table 1 contains estimates of h for the Euler scheme (7) and the new scheme (8), where ε is the desirable error of the scheme $E|x_T-x_K|$.

Table 1. Discretization steps h.

a	-1	-1	1	1
ε	0.001	0.005	0.001	0.005
Euler scheme	0.0020	0.0098	0.00035	0.0017
new scheme	0.0036	0.0182	0.00169	0.0055

4. SIMULATION RESULTS

We present, in this section, results of numerical experiments. We considered the scalar SDE (9)

$$dx_t = ax_t + b\,dw_t.$$

After the computation of the quadratic mean errors $E\big(x_T - x_K^e\big)^2$, $E\big(x_T - x_K^b\big)^2$ by the Monte Carlo method, we estimated the unknown coefficients C_e, C_b for schemes (7), (8) by the least square method. We estimated $E\big(x_T - x_K^e\big)^2$, $E\big(x_T - x_K^b\big)^2$ by

$$\mathcal{E}_e = \frac{1}{100}\sum_{i=1}^{100}\left((x_K)_i - (x_K^e)_i\right)^2, \quad \mathcal{E}_b = \frac{1}{100}\sum_{i=1}^{100}\left((x_K)_i - (x_K^b)_i\right)^2,$$

where index i corresponds to the i^{th} realization.

Tables 2 and 3 contain values of $\mathcal{E}_e$ and $\mathcal{E}_b$ for three different h and show that scheme (8) gives in mean smaller error. On the basis of these tables, we estimated C_e and C_b by LS method: we have for $a = -1$: $C_e = 0.296$, $C_b = 0.003$, and for $a = 1$: $C_e = 2.361$, $C_b = 0.034$. Mention should be made that these estimates of C_e, C_b are not in keeping with the estimates of the example from the previous section, simulation results give smaller estimations mainly for the new scheme (8).

Table 2. $a = -1,\ b = 1,\ x_0 \approx \mathcal{N}(0, 0.5),\ T = 1.2$.

h	$\mathcal{E}_e$	$\mathcal{E}_b$
0.01	0.00000724	0.00000000001
0.05	0.00019259	0.00000000444
0.1	0.00088011	0.00000007325

Table 3. $a = 1,\ b = 1,\ x_0 \approx \mathcal{N}(0, 0.5),\ T = 1.2$.

h	$\mathcal{E}_e$	$\mathcal{E}_b$
0.01	0.000717	0.0000000014
0.05	0.014824	0.0000007701
0.1	0.055528	0.0000121558

5. RUNGE-KUTTA SCHEME

The Stratonovich SDE

$$dX_t = F(t, X_t)\,dt + G(t, X_t) \circ dW_t \tag{12}$$

with initial value X_0 and $t \in [0, T]$, where

$$\begin{aligned} X_t &= \left(x_t^1, x_t^2, \dots, x_t^p\right)', \\ F(t, X_t) &= \left(f_{(t,X_k)}^1, f_{(t,X_k)}^2, \dots, f_{(t,X_k)}^p\right)', \\ G(t, X_t) &= \left\{g_{(t,X_k)}^{ij}\right\}_{i=\overline{1,p},\, j=\overline{1,q}}, \qquad \text{and} \\ W_t &= \left(w_t^1, w_t^2, \dots, w_t^q\right)', \end{aligned}$$

express in a form

$$dx^i(t) = f^i(t, X_t)\,dt + \sum_{j=1}^{q} g^{ij}(t, X_t) \circ dw^j(t), \qquad i = 1, 2, \dots, p.$$

Let $t_k = kh$, $k = 0, 1, \dots, K$, where the discretization step $h = T/K$. Consider the iterative scheme of the Runge-Kutta method of order M

$$\left.\begin{aligned} \tilde{x}_0^i &= x_0^i\,, \\ \tilde{x}_{k+1}^i &= \tilde{x}_k^i + \sum_{l=1}^{M} \gamma_l \left(h f_{kl}^i + \sum_{j=1}^{q} g_{kl}^{ij}\,\Delta w_{k+1}^j \right), \end{aligned}\right\} \qquad \begin{aligned} k &= 0, 1, \dots, K-1, \\ i &= 1, 2, \dots, p, \end{aligned} \tag{13}$$

where

$$\Delta w_{k+1}^j = w_{t_{k+1}}^j - w_{t_k}^j, \quad f_{k1}^i = f^i(t_k, \tilde{X}_k),$$
$$g_{k1}^{ij} = g^{ij}(t_k, \tilde{X}_k), \quad \tilde{X}_{k1} \equiv \tilde{X}_k = (\tilde{x}_k^1, \tilde{x}_k^2, \dots, \tilde{x}_k^n)',$$

$t_{k1} \equiv t_k$ and for $l = 2, 3, \dots, M$

$$f_{kl}^i = f^i(t_{kl}, \tilde{X}_{kl}), \quad g_{kl}^{ij} = g^{ij}(t_{kl}, \tilde{X}_{kl}),$$

here $t_{kl} = t_k + h\alpha_l$, and $\tilde{X}_{kl}$ is a vector, the i^{th} component of which equals

$$\tilde{x}_k^i + \sum_{L=1}^{l-1} \beta_{lL} \left(h f_{kL}^i + \sum_{j=1}^{q} g_{kL}^{ij}\,\Delta w_k^j \right),$$

and the parameters α_l, β_{lL}, γ_l correspond to the parameters of the Runge-Kutta method.

From the paper by Rümelin [5], it follows that the iterative scheme (13) converges in quadratic mean (as regards the conditions of convergence see Theorem 1 in [5]) to the strong solution of the Ito SDE

$$dX_t = \left(F(t, X_t) + \lambda \sum_{j=1}^{q} (\nabla_x G^j) G^j \right) dt + G(t, X_t)\,dW_t,$$

where

$$\lambda = \begin{cases} 0, & \text{if } M = 1, \\ \frac{1}{2}, & \text{if } M \geq 2, \end{cases}$$

G^j denotes the j^{th} column of matrix G and

$$(\nabla_x G^j) = \left\{ \frac{\partial}{\partial x_m} g^{i,j} \right\}_{i,m=\overline{1,p}}.$$

We must mention that in [5], $\lambda \neq 1/2$ for $M \geq 2$ but

$$\lambda = \sum_{l=2}^{M} \gamma_l \sum_{L=1}^{l-1} \beta_{lL}.$$

By [5], λ may be equal to any real number (for $M \geq 2$), however, we show that $\lambda = 1/2$ independently from values γ_l and β_{lL}. Indeed, for parameters α, β, γ the following equations hold (see, e.g., [6])

$$\sum_{l=2}^{M} \gamma_l\,\alpha_l = \frac{1}{2},$$

$$\alpha_l = \sum_{L=1}^{l-1} \beta_{lL}, \qquad l = 2, 3, \dots, M.$$

Hence, we get that $\lambda = 1/2$ for $M \geq 2$.

We observe that the vector form of (13) is

$$\left.\begin{aligned}
&\tilde{X}_0 = X_0,\\
&\tilde{X}_{k+1} = \tilde{X}_k + \sum_{l=1}^{M} \gamma_l \left(hF\big(t_{kl}, \tilde{X}_{kl}\big) + G\big(t_{kl}, \tilde{X}_{kl}\big)\Delta W_{k+1} \right),
\end{aligned}\right\} \qquad k = 0, 1, \dots, K-1, \tag{14}$$

where

$$\tilde{X}_k = \left(\tilde{x}_k^1, \tilde{x}_k^2, \dots, \tilde{x}_k^n\right)', \quad \Delta W_{k+1} = W_{t_{k+1}} - W_{t_k},$$

$t_{k1} = t_k$, $\tilde{X}_{k1} = \tilde{X}_k$ and for $l = 2, 3, \dots, M$

$$\begin{aligned}
t_{kl} &= t_k + h\alpha_l,\\
\tilde{X}_{kl} &= \tilde{X}_k + \sum_{L=1}^{l-1} \beta_{lL} \left(hF\big(t_{kL}, \tilde{X}_{kL}\big) + G\big(t_{kL}, \tilde{X}_{kL}\big)\Delta W_k \right).
\end{aligned}$$

Let $F(t, X_t) = AX_t$, where $A = \{a_{ij}\}_{i,j=\overline{1,p}}$ and $G(t, X_t) = B = \{b_{ij}\}_{i=\overline{1,p}, j=\overline{1,q}}$, i.e., consider linear SDE with constant coefficients (1)

$$dX_t = AX_t\, dt + B\, dW_t \equiv AX_t\, dt + B \circ dW_t.$$

After arrangements in (13), the Runge-Kutta scheme (14) reduces to the simple form

$$\left.\begin{aligned}
&\tilde{X}_0 = X_0,\\
&\tilde{X}_{k+1} = Q\tilde{X}_k + P\Delta W_{k+1},
\end{aligned}\right\} \qquad k = 0, 1, \dots, K-1, \tag{15}$$

where

$$\begin{aligned}
Q &= I + \sum_{l=1}^{M} c_l \big(hA\big)^l,\\
P &= \sum_{l=1}^{M} c_l \big(hA\big)^{l-1} B,
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= \sum_{l_1=1}^{M} \gamma_{l_1} \equiv 1,\\
c_2 &= \sum_{l_1=2}^{M} \sum_{l_2=1}^{l_1-1} \gamma_{l_1} \beta_{l_1 l_2},\\
&\vdots\\
c_M &= \sum_{l_1=M}^{M} \sum_{l_2=M-1}^{l_1-1} \cdots \sum_{l_{M-1}=2}^{l_{M-2}-1} \sum_{l_M=1}^{l_{M-1}-1} \gamma_{l_1} \beta_{l_1 l_2} \dots \beta_{l_{M-2} l_{M-1}} \beta_{l_{M-1} l_M}.
\end{aligned}$$

From (15) for $M = 1$, $\gamma_1 = 1$, we get the Euler scheme (6), for $M = 2$, $\gamma_1 = \gamma_2 = 0.5$ and $\beta_{21} = 1$, we obtain the Heun scheme

$$\tilde{X}_{k+1} = \left(I + hA + \frac{h^2}{2}A^2 \right) \tilde{X}_k + \left(I + \frac{h}{2}A \right) B\Delta W_{k+1},$$

and for $M = 4$ with the well known Runge-Kutta coefficients, we have

$$\begin{aligned}\tilde{X}_{k+1} &= \left(I + \sum_{l=1}^{4} \frac{1}{l!}(hA)^l\right)\tilde{X}_k + \left(\sum_{l=1}^{4}\frac{1}{l!}(hA)^{l-1}\right)B\Delta W_{k+1} \\ &= \left(I + hA + \frac{h^2}{2}A^2 + \frac{h^3}{3!}A^3 + \frac{h^4}{4!}A^4\right)\tilde{X}_k \\ &\quad + \left(I + \frac{h}{2}A + \frac{h^2}{3!}A^2 + \frac{h^3}{4!}A^3\right)B\Delta W_{k+1}.\end{aligned}$$

From (15), it also follows that the numerical solution of X_T by the Runge-Kutta method of order M equals

$$\tilde{X}_K = Q^K X_0 + \sum_{k=0}^{K-1} Q^{K-k-1} P \Delta W_k,$$

whence for $M = 1$, $\gamma_1 = 1$ we get (7).

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